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Some Extremal Properties of Perfect Splines and the Pointwise Landau Problem on the Finite Interval*

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1. INTRODUCTION

Let $W_\infty^{(n)} = W_\infty^{(n)}[0, 1] = \{f: f \in C^{n-1}[0, 1], f^{(n-1)}$ absolutely continuous, $f^{(n)} \in L^\infty[0, 1]\}$. In this paper we consider the extremum problem

$$\sup\{| \lambda f^{(k)}(\xi) + \mu f^{(k-1)}(\xi)| : f \in W_\infty^{(n)}, \|f\|_\infty \leqslant 1, \|f^{(n)}\|_\infty \leqslant \sigma\}, \quad (1.1)$$

where $\xi \in [0, 1]$, $1 \leqslant k \leqslant n-1$, and λ, μ real, are all fixed, and $\|\cdot\|_\infty$ denotes the usual L^∞ norm on [0, 1].

We prove that in the discussion of (1.1), it is sufficient to consider a specific class $\mathscr{P}(\sigma)$ of perfect splines $P(x)$ of degree n which satisfy $\|P\|_\infty = 1$ and $\|P^{(n)}\|_\infty = \sigma$, and certain more restrictive requirements as stipulated in Theorem 5.1. We also consider in more detail a special case of (1.1), viz.,

$$\sup\{|f^{(k)}(\xi)| : f \in W_\infty^{(n)}, \|f\|_\infty \leqslant 1, \|f^{(n)}\|_\infty \leqslant \sigma\}, \quad (1.2)$$

with $\xi \in [0, 1]$, $1 \leqslant k \leqslant n-1$, fixed. The extremum problem (1.2) may be regarded as a pointwise version of the Landau problem on the finite interval given by

$$\sup\{\|f^{(k)}\|_\infty : f \in W_\infty^{(n)}, \|f\|_\infty \leqslant 1, \|f^{(n)}\|_\infty \leqslant \sigma\}. \quad (1.3)$$

(For a discussion of the Landau problem, see Schoenberg [17] and the references therein.) By construction of numerical differentiation formulas, we show that for each k, $1 \leqslant k \leqslant n-1$, fixed, every element of $\mathscr{P}(\sigma)$ achieves the maximum in (1.2) for at least one $\xi \in [0, 1]$.

Our results may also be viewed as an extension of the pointwise V. A. Markov inequalities for polynomials, to the appropriate Sobolev space (cf. the work of Gusev in [20, pp. 179–197]).

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The organization of this paper runs as follows. Sections 2 and 3 are preliminary sections where we list some known but, unfortunately, not sufficiently well-known properties of perfect splines and generalized perfect splines. In Section 4, we show that in the consideration of the extremum problem (1.1), it is sufficient to consider perfect splines with at most a finite number of knots, the number depending on σ, and which exhibit certain equioscillation properties. In Section 5, we prove the main result (Theorem 5.1). Section 6 is a discussion of exact numerical differentiation formulas on [0, 1], and relates to (1.2).

2. Preliminaries: Perfect Splines

In this section, we list several properties of perfect splines which shall prove useful in the succeeding sections.

Definition 2.1. A perfect spline on [0, 1] of degree n with r knots $\{\xi_i\}_{i=1}^r$, $0 < \xi_1 < \cdots < \xi_r < 1$, is any function $P(x)$ of the form

$$P(x) = \sum_{i=0}^{n-1} a_i x^i + c\left[x^n + 2\sum_{j=1}^{r} (-1)^j (x-\xi_j)_+^n\right], \tag{2.1}$$

where, as usual, $x_+{}^n = x^n$ if $x \geqslant 0$, and zero otherwise.

It is, at times, more convenient to write $P(x)$ in the equivalent form

$$P(x) = \sum_{i=0}^{n-1} a_i x^i + nc \sum_{j=0}^{r} (-1)^j \int_{\xi_j}^{\xi_{j+1}} (x-t)_+^{n-1}\, dt, \tag{2.2}$$

where $\xi_0 = 0$, $\xi_{r+1} = 1$.

If $c = 0$, then we say that $P(x)$ is a perfect spline of degree n with -1 knots.

In what follows, all perfect splines under consideration are of degree n, and we thus delete all reference to the degree of the perfect spline.

Proposition 2.1. *Any nontrivial perfect spline with exactly r knots has at most $n + r$ zeros, counting multiplicity.*

We count the multiplicity of a zero of a perfect spline in the following manner:

If x_0 is not a knot of $P(x)$, then we count multiplicity in the usual manner, i.e., $P(x)$ has a zero of multiplicity m at x_0 iff $P^{(i)}(x_0) = 0$, $i = 0, 1,..., m-1$, and $P^{(m)}(x_0) \neq 0$. If x_0 is a knot of $P(x)$, then $P(x)$ has a zero of multiplicity m at x_0, $m \leqslant n-1$, iff $P^{(i)}(x_0) = 0$, $i = 0, 1,..., m-1$, and $P^{(m)}(x_0) \neq 0$. If $P^{(i)}(x_0) = 0$, $i = 0, 1,..., n-1$, then since $P^{(n)}(x_0+)\, P^{(n)}(x_0-) < 0$, we say that $P(x)$ has a zero of multiplicity n at x_0.

The proof of Proposition 2.1 is a simple consequence of Rolle's theorem, see Cavaretta [4] and Karlin [8].

The converse to Proposition 2.1 is contained in the following result.

PROPOSITION 2.2. *Given* $\{x_i\}_{i=1}^{n+r}$, $0 \leqslant x_1 \leqslant \cdots \leqslant x_{n+r} \leqslant 1$, *with* $x_i < x_{i+n}$, $i = 1,\ldots, r$. *Then there exists a nontrivial perfect spline* $P(x)$, *unique up to multiplication by a constant, with exactly* r *knots such that* $P(x_i) = 0$, $i = 1,\ldots, n + r$. *Furthermore, if* $\{\xi_i\}_{i=1}^{r}$ *are the knots of* $P(x)$, *then*

$$x_i < \xi_i < x_{i+1}, \qquad i = 1,\ldots, r. \tag{2.3}$$

Remark 2.1. If $f \in W_\infty^{(n)}$ and $x_{i-1} < x_i = x_{i+1} = \cdots = x_{i+m-1} < x_{i+m}$, $m \leqslant n$, then by $f(x_j) = 0$ for $j = i,\ldots, i + m - 1$, we mean $f^{(l)}(x_i) = 0$, $l = 0, 1,\ldots, m - 1$.

For a proof of Proposition 2.2, see Cavaretta [4] and Karlin [8].

THEOREM 2.1. *If* $g \in W_\infty^{(n)}$, *and* $\{x_i\}_{i=1}^{n+r+1}$ *are given,* $0 \leqslant x_1 \leqslant \cdots \leqslant x_{n+r+1} \leqslant 1$, *with* $x_i < x_{i+n}$, $i = 1,\ldots, r + 1$, *then there exists a perfect spline* $P(x)$ *with at most* r *knots such that* $P(x_i) = g(x_i)$, $i = 1,\ldots, n + r + 1$. (*In case of equality among the* x_i, *see Remark* 2.1.) *Furthermore, for any perfect spline with at most r knots satisfying the above interpolating conditions,*

$$\| P^{(n)} \|_\infty \leqslant \| g^{(n)} \|_\infty . \tag{2.4}$$

Theorem 2.1 gives one of the essential extremizing properties of perfect splines. For a proof of the above theorem, see de Boor [1] and Karlin [8].

THEOREM 2.2. *Let* $P(x)$ *be any perfect spline with at most* r *knots and let* $\{x_i\}_{i=1}^{n+r+1}$ *be given, as above. Set* $\sigma^* = \| P^{(n)} \|_\infty$. *Then for each* $\sigma > \sigma^*$ *there exist two unique perfect splines* $\bar{P}_\sigma(x)$ *and* $\underline{P}_\sigma(x)$, *each with exactly* $r + 1$ *knots, satisfying*

1. $\bar{P}_\sigma(x_i) = \underline{P}_\sigma(x_i) = P(x_i)$, $i = 1,\ldots, n + r + 1$,
2. $\| \bar{P}_\sigma^{(n)} \|_\infty = \| \underline{P}_\sigma^{(n)} \|_\infty = \sigma$,
3. $\bar{P}_\sigma^{(n)}(1) = \sigma$, $\underline{P}_\sigma^{(n)}(1) = -\sigma$.

This theorem and applications thereof may be found in Micchelli and Miranker [16], Gaffney and Powell [5], and de Boor [2]. The proof is essentially due to Krein [13, 14], see also Karlin and Studden [12, p. 263].

To fix our notation, the following definition shall hold throughout this paper.

DEFINITION 2.2. We say that a function $f \in C[0, 1]$ has l points of equioscillation if there exist $\{x_i\}_{i=1}^{l}$, $0 \leqslant x_1 < \cdots < x_l \leqslant 1$, and $\epsilon \in \{-1, 1\}$ such that $f(x_i)(-1)^i \epsilon = \|f\|_\infty$, $i = 1,\ldots, l$.

THEOREM 2.3. *For each integer $r \geqslant -1$, there exists a perfect spline $P_r(x)$, unique up to multiplication by -1, with exactly r knots such that $\| P_r \|_\infty = 1$, and $P_r(x)$ equioscillates at $n + r + 1$ points in* $[0, 1]$.

For a proof of this theorem, see Tihomirov [19], Karlin [9], and Cavaretta [4]. Note that $P_0 = T_n$, the Chebyshev polynomial of degree n, and $P_{-1} = T_{n-1}$.

Let $\sigma_r = \| P_r^{(n)} \|_\infty$. Then, from Karlin [9], we have

PROPOSITION 2.3. $\sigma_{-1} = 0 < \sigma_0 < \sigma_1 < \cdots$, *and* $\sigma_r \uparrow \infty$ *as* $r \uparrow \infty$.

THEOREM 2.4. *For $\sigma \in (\sigma_r, \sigma_{r+1})$ there exists a perfect spline $Z(x; \sigma)$ with exactly $r + 1$ knots and $n + r + 1$ points of equioscillation such that $\| Z(\cdot; \sigma)\|_\infty = 1$, and $\| Z^{(n)}(\cdot; \sigma)\|_\infty = \sigma$. Furthermore, if $P(x)$ is any perfect spline with the above properties, i.e., having at most $r + 1$ knots, at least $n + r + 1$ points of equioscillation, and $\| P \|_\infty = 1$, $\| P^{(n)} \|_\infty = \sigma$, then $P(x) = \pm Z(x; \sigma)$ or $P(x) = \pm Z(1 - x; \sigma)$.*

A proof of the above result and a thorough discussion of this class of perfect splines is to be found in Karlin [9]. These perfect splines $Z(x; \sigma)$ are called Zolotarev perfect splines and $Z(x; \sigma)$ may be uniquely determined by the normalization

(i) $Z(1; \sigma) = 1$,

(ii) $Z^{(n)}(1; \sigma) = \sigma$.

We define $Z(x; \sigma)$ for $\sigma = \sigma_r$ as $P_r(x)$.

3. PRELIMINARIES: GENERALIZED PERFECT SPLINES

Due to difficulties which arise in working with perefct splines, we introduce a particular notion of a generalized perfect spline and record its various properties. The use of generalized perfect splines has previously appeared in various contexts, cf. Karlin [8–10] and de Boor [1].

Rather than considering the Sobolev space $W_\infty^{(n)}$, i.e., functions $f(x)$ on $[0, 1]$ of the form

$$f(x) = \sum_{i=0}^{n-1} a_i x^i + \frac{1}{(n-1)!} \int_0^1 (x - t)_+^{n-1} h(t)\, dt,$$

where $h(t) = f^{(n)}(t) \in L^\infty[0, 1]$, we define the functions

$$\begin{aligned} u_i(x; \epsilon) &= \frac{1}{(2\pi\epsilon)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\eta)^2/2\epsilon} \eta^i \, d\eta, \qquad i = 0, 1, \ldots, n-1, \\ K(x, t; \epsilon) &= \frac{1}{(2\pi\epsilon)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\eta)^2/2\epsilon} (\eta - t)_+^{n-1} \, d\eta, \end{aligned} \tag{3.1}$$

for $\epsilon > 0$, and consider the space

$$W_\infty^{(n)}(\mathscr{K}_\epsilon) = \left\{ f: f(x) = \sum_{i=0}^{n-1} a_i u_i(x; \epsilon) + \frac{1}{(n-1)!} \int_0^1 K(x, t; \epsilon)\, h(t)\, dt, \right.$$

$$\left. \text{where } h \in L^\infty[0, 1] \right\}.$$

For ease of notation, we set $(Nf)(x) = h(x)$, $x \in [0, 1]$. For $\epsilon = 0$, this reduces to $Nf(x) = f^{(n)}(x)$. The advantages in dealing with $W_\infty^{(n)}(\mathscr{K}_\epsilon)$ rather than $W_\infty^{(n)}$ will become obvious from this and the succeeding section.

It is important to note that $u_i(x; \epsilon) \to x^i$, $i = 0, 1,..., n-1$, and $K(x, t; \epsilon) \to (x - t)_+^{n-1}$, as $\epsilon \downarrow 0$, uniformly on $[0, 1]$ (and $[0, 1] \times [0, 1]$).

In general, when dealing with an extended complete Chebyshev system $\{u_i(x)\}_{i=0}^{n-1}$, one replaces the natural derivatives d^k/dx^k by $D_k \cdots D_1$, where D_l is a first-order differential operator obtained from a factorization of $\{u_i(x)\}_{i=0}^{n-1}$; cf. Karlin [7, Chap. 10]. However $u_i(x; \epsilon)$ is a monic polynomial of degree i, and thus the natural derivatives d^k/dx^k, $1 \leqslant k \leqslant n-1$, are maintained in our case. We now restate the results of Section 2 for generlaized perfect splines. The proofs of these results are either contained in the references of the previous section or are variants of the proofs found therein.

DEFINITION 3.1. We say that $P(x; \epsilon)$ is a generalized perfect spline (G.P.S.) with r knots, if there exists $\{\xi_i\}_{i=1}^r$, $\xi_0 = 0 < \xi_1 < \cdots < \xi_r < 1 = \xi_{r+1}$, such that

$$P(x; \epsilon) = \sum_{i=0}^{n-1} a_i u_i(x; \epsilon) + c \sum_{i=0}^{r} (-1)^i \int_{\xi_i}^{\xi_{i+1}} K(x, t; \epsilon)\, dt. \tag{3.2}$$

As in Section 2, we have dropped all reference to the degree of the G.P.S. For ease of notation, we shall also suppress the ϵ throughout this and the next sections in the case where no ambiguity arises.

PROPOSITION 3.1. *Any nontrivial* G.P.S. *with exactly r knots has at most $n + r$ zeros, counting multiplicity.*

PROPOSITION 3.2. *Given $\{x_i\}_{i=1}^{n+r}$, $0 \leqslant x_1 \leqslant \cdots \leqslant x_{n+r} \leqslant 1$, then there exists a nontrivial* G.P.S. *$P(x)$, unique up to a multiplicative constant, with exactly r knots such that $P(x_i) = 0$, $i = 1,..., n + r$.*

An important difference between perfect splines and generalized perfect splines is that no restriction of the form $x_i < x_{i+n}$, $i = 1,..., r$, is made for generalized perfect splines.

THEOREM 3.1. *If $f \in W_\infty^{(n)}(\mathcal{K}_\epsilon)$, and $\{x_i\}_{i=1}^{n+r+1}$ are given, $0 \leqslant x_1 \leqslant \cdots \leqslant x_{n+r+1} \leqslant 1$, then there exists a unique* G.P.S. *$P(x)$, with at most r knots, such that $P(x_i) = f(x_i)$, $i = 1,\ldots, n + r + 1$. Furthermore, $\| N(P)\|_\infty \leqslant \| N(f)\|_\infty$.*

Note that uniqueness obtains here but not in Theorem 2.1.

It is important to note that we may replace some of the Hermite data given in Theorem 3.1 by even block data. This fact is of crucial importance in the next section. We shall make use of the following variant of Theorem 3.1.

THEOREM 3.1(a). *If $f \in W_\infty^{(n)}(\mathcal{K}_\epsilon)$, and $\{x_i\}_{i=1}^{n+r+1}$, ξ and k are given, $0 \leqslant x_1 \leqslant \cdots \leqslant x_{n+r-1} \leqslant 1$, $\xi \in [0, 1]$, $1 \leqslant k \leqslant n - 1$, then there exists a unique* G.P.S. *$P(x)$ with at most r knots such that $P(x_i) = f(x_i)$, $i = 1,\ldots, n + r - 1$, and $P^{(k)}(\xi) = f^{(k)}(\xi)$, $P^{(k-1)}(\xi) = f^{(k-1)}(\xi)$. Furthermore, $\| N(P)\|_\infty \leqslant \| N(f)\|_\infty$.*

The interpolatory conditions are meant as stated except when, for some i and some $m \geqslant k$, $x_{i-1} < x_i = \cdots = x_{i+m-1} = \xi < x_{i+m}$, in which case we demand that $P^{(l)}(x_i) = f^{(l)}(x_i)$ for $i = 0, 1,\ldots, m + 1$.

THEOREM 3.2. *Let $P(x)$ be any* G.P.S. *with at most r knots, and let $\{x_i\}_{i=1}^{n+r+1}$, $0 \leqslant x_1 \leqslant \cdots \leqslant x_{n+r+1} \leqslant 1$ be given. Set $\sigma^* = \| N(P)\|_\infty$. Then for each $\sigma > \sigma^*$, there exist two unique* G.P.S.'s *$\bar{P}_\sigma(x)$ and $\underline{P}_\sigma(x)$, each with exactly $r + 1$ knots, satisfying*

1. $\bar{P}_\sigma(x_i) = \underline{P}_\sigma(x_i) = P(x_i)$, $i = 1,\ldots, n + r + 1$,
2. $\| N(\bar{P}_\sigma)\|_\infty = \| N(\underline{P}_\sigma)\|_\infty = \sigma$,
3. $N(\bar{P}_\sigma(1)) = \sigma$, $N(\underline{P}_\sigma(1)) = -\sigma$.

We state the following extension of Theorem 3.2, similar to Theorem 3.1(a).

THEOREM 3.2(a). *Let $P(x)$ be any* G.P.S. *with at most r knots, and let $\{x_i\}_{i=1}^{n+r-1}$, $0 \leqslant x_1 \leqslant \cdots \leqslant x_{n+r-1} \leqslant 1$, $\xi \in [0, 1]$, and k, $1 \leqslant k \leqslant n - 1$, be given. Assume $\| N(P)\|_\infty = \sigma^*$. Then for each $\sigma > \sigma^*$, there exist two unique* G.P.S.'s *$\bar{P}_\sigma(x)$ and $\underline{P}_\sigma(x)$, each with exactly $r + 1$ knots, satisfying*

1. $\bar{P}_\sigma(x_i) = \underline{P}_\sigma(x_i) = P(x_i)$, $i = 1,\ldots, n + r - 1$,
2. $\bar{P}_\sigma^{(j)}(\xi) = \underline{P}_\sigma^{(j)}(\xi) = P^{(j)}(\xi)$, $j = k - 1, k$,
3. $\| N(\bar{P}_\sigma)\|_\infty = \| N(\underline{P}_\sigma)\|_\infty = \sigma$,
4. $N(\bar{P}_\sigma(1)) = \sigma$, $N(\underline{P}_\sigma(1)) = -\sigma$.

THEOREM 3.3. *For each integer $r \geqslant 0$, there exists a* G.P.S. *$P_r(x)$, unique up to multiplication by -1, with exactly r knots such that $\| P_r \|_\infty = 1$, and $P_r(x)$ equioscillates between 1 and -1 at exactly $n + r + 1$ points in $[0, 1]$.*

Set $\sigma_r = \| N(P_r)\|_\infty$.

PROPOSITION 3.3. *$\sigma_{-1} = 0 < \sigma_0 < \sigma_1 < \cdots < \sigma_r < \cdots$ and $\sigma_r \uparrow \infty$ as $r \uparrow \infty$.*

THEOREM 3.4. *For $\sigma \in (\sigma_r, \sigma_{r+1})$, there exist two* G.P.S.'s *$Z_i(x; \sigma)$, $i = 1, 2$, unique up to multiplication by -1, with exactly $r + 1$ knots and $n + r$ points of equioscillation such that $\| Z_i(\cdot; \sigma)\|_\infty = 1$ and $\| NZ_i(\cdot; \sigma)\|_\infty = \sigma$, $i = 1, 2$. Let $\{x_j^i\}_{j=1}^{n+r+1}$, $0 \leqslant x_1^i < \cdots < x_{n+r+1}^i \leqslant 1$ denote the points of equioscillation of $Z_i(x; \sigma)$, $i = 1, 2$. Then, $Z_1(x; \sigma)$ and $Z_2(x; \sigma)$ are differentiated by the fact that*

$$(NZ_1(1; \sigma))(Z_1(x^1_{n+r+1}; \sigma)) = \sigma, \text{ while } (NZ_2(1; \sigma))(Z_2(x^2_{n+r+1}; \sigma)) = -\sigma.$$

4. PERFECT SPLINES ARE SUFFICIENT

Set $W_\infty^{(n)}(\mathscr{K}_\epsilon; \sigma) = \{f : f \in W_\infty^{(n)}(\mathscr{K}_\epsilon), \|f\|_\infty \leqslant 1, \| N(f)\|_\infty \leqslant \sigma\}$.
We are interested in the problem

$$\sup_{f \in W_\infty^{(n)}(\mathscr{K}_\epsilon; \sigma)} | \lambda f^{(k)}(\xi) + \mu f^{(k-1)}(\xi)| \tag{4.1}$$

for some $\xi \in [0, 1]$, $1 \leqslant k \leqslant n - 1$, fixed, and λ, μ real numbers.

We first show that in the study of (4.1) it is sufficient to consider generalized perfect splines with at most a finite number (this number is dependent on σ) of knots.

THEOREM 4.1. *If $\sigma \in [\sigma_r, \sigma_{r+1})$, then for any $f \in W_\infty^{(n)}(\mathscr{K}_\epsilon; \sigma)$, there exists a* G.P.S. *$P(x)$ with at most $r + 4$ knots such that $P \in W_\infty^{(n)}(\mathscr{K}_\epsilon; \sigma)$, and $P^{(j)}(\xi) = f^{(j)}(\xi), j = k - 1, k$.*

We shall not consider the case where $\sigma = 0$. In this and the succeeding sections, if $r = -1$, then we understand $\sigma \in [\sigma_{-1}, \sigma_0)$ to mean $\sigma \in (\sigma_{-1}, \sigma_0)$.
In the proof of Theorem 4.1, we make use of the following simple lemma.

LEMMA 4.1. *If $P(x)$ is a* G.P.S. *with $r + 4$ knots, $f \in W_\infty^{(n)}(\mathscr{K}_\epsilon)$, and there exist $\{x_i\}_{i=1}^{n+r+2}$, $0 \leqslant x_1 \leqslant \cdots \leqslant x_{n+r+2} \leqslant 1$, and $\xi \in [0, 1]$ such that*

(1) $P(x_i) = f(x_i)$, $i = 1,..., n + r + 2$,
(2) $P^{(j)}(\xi) = f^{(j)}(\xi)$, $j = k - 1, k$, *and*
(3) $\| N(P)\|_\infty > \| N(f)\|_\infty$,

then $P(x) - f(x)$ has no additional zeros in $[0, 1]$.

Proof. If $P(x) - f(x)$ has an additional zero, counting multiplicity, then by Theorem 3.1(a), $\| N(P)\|_\infty \leqslant \| N(f)\|_\infty$. But this contradicts (3). The lemma is proved.

Proof of Theorem 4.1. Let $Z = \{\boldsymbol{\zeta} = (\zeta_1, ..., \zeta_{n+r+3}): \zeta_i \geqslant 0, \sum_{i=1}^{n+r+3} \zeta_i = 1\}$. Let $x_0(\boldsymbol{\zeta}) = 0$, $x_i(\boldsymbol{\zeta}) = \sum_{j=1}^{i} \zeta_j$, $i = 1,..., n + r + 3$, $(x_{n+r+3}(\boldsymbol{\zeta}) = 1)$, and assume $f \in W_\infty^{(n)}(\mathscr{K}_\epsilon; \sigma)$. From Theorem 3.1(a), there exists, for each $\boldsymbol{\zeta} \in Z$, a unique G.P.S. $\tilde{P}(x; \boldsymbol{\zeta})$ with at most $r + 3$ knots such that

(1) $\tilde{P}(x_i(\boldsymbol{\zeta}); \boldsymbol{\zeta}) = f(x_i(\boldsymbol{\zeta}))$, $i = 1,..., n + r + 2$.

(2) $\tilde{P}^{(j)}(\xi; \boldsymbol{\zeta}) = f^{(j)}(\xi)$, $j = k - 1, k$

and furthermore, $\| N(\tilde{P}(\cdot; \boldsymbol{\zeta}))\|_\infty \leqslant \| N(f)\|_\infty \leqslant \sigma$. Choose $\delta > 0$, sufficiently small, such that $\sigma + \delta < \sigma_{r+1}$. From Theorem 3.2(a), there exists a unique G.P.S. $P_\delta(x; \boldsymbol{\zeta})$ with exactly $r + 4$ knots satisfying (1), (2), and

(3) $\| N(P_\delta(\cdot; \boldsymbol{\zeta}))\|_\infty = \sigma + \delta$, and $N(P_\delta(1; \boldsymbol{\zeta})) = \sigma + \delta$.

Define

$$m_i(\boldsymbol{\zeta}) = \max_{x_{i-1}(\boldsymbol{\zeta}) \leqslant x \leqslant x_i(\boldsymbol{\zeta})} | P_\delta(x; \boldsymbol{\zeta})|, \qquad i = 1,..., n + r + 3,$$

$$M(\boldsymbol{\zeta}) = \max_{i=1,...,n+r+3} m_i(\boldsymbol{\zeta}) = \| P_\delta(\cdot; \boldsymbol{\zeta})\|_\infty,$$

and $R_i(\boldsymbol{\zeta}) = M(\boldsymbol{\zeta}) - m_i(\boldsymbol{\zeta})$, $i = 1,..., n + r + 3$. Note that $R_i(\boldsymbol{\zeta})$ is a continuous function of $\boldsymbol{\zeta} \in Z$.

It is our aim to prove the existence of a $\boldsymbol{\zeta}^* \in Z$ for which $\| P_\delta(\cdot; \boldsymbol{\zeta}^*)\|_\infty \leqslant 1$. The proof of this fact is divided into three cases.

Case 1. There exists a $\boldsymbol{\zeta}^* \in Z$ such that $\sum_{i=1}^{n+r+3} R_i(\boldsymbol{\zeta}^*) = 0$, and $\zeta_i^* = 0$ for some $i = 1,..., n + r + 3$.

Since $\sum_{i=1}^{n+r+3} R_i(\boldsymbol{\zeta}^*) = 0$, $M(\boldsymbol{\zeta}^*) = m_i(\boldsymbol{\zeta}^*)$ for all $i = 1,..., n + r + 3$. Now, for some i_0, $\zeta_{i_0}^* = 0$, implying $m_{i_0}(\boldsymbol{\zeta}^*) = | P_\delta(x_{i_0}(\boldsymbol{\zeta}^*); \boldsymbol{\zeta}^*)| = | f(x_{i_0}(\boldsymbol{\zeta}^*))| \leqslant 1$. Thus $\| P_\delta(\cdot; \boldsymbol{\zeta}^*)\|_\infty = M(\boldsymbol{\zeta}^*) = m_{i_0}(\boldsymbol{\zeta}^*) \leqslant 1$.

Case 2. There exists a $\boldsymbol{\zeta}^* \in Z$ such that $\sum_{i=1}^{n+r+3} R_i(\boldsymbol{\zeta}^*) = 0$, and $\zeta_i^* > 0$, $i = 1,..., n + r + 3$.

Since $M(\boldsymbol{\zeta}^*) = \| P_\delta(\cdot; \boldsymbol{\zeta}^*)\|_\infty$, if $M(\boldsymbol{\zeta}^*) \leqslant 1$, then we are finished. Assume $M(\boldsymbol{\zeta}^*) = c > 1$. Then in each interval $(x_{i-1}(\boldsymbol{\zeta}^*), x_i(\boldsymbol{\zeta}^*))$, there exists a point z_i, $i = 1,..., n + r + 3$, such that $| P_\delta(z_i; \boldsymbol{\zeta}^*)| = c$. From Lemma 4.1, since $\| f\|_\infty \leqslant 1$, it follows that $P_\delta(z_i; \boldsymbol{\zeta}^*) P_\delta(z_{i+1}; \boldsymbol{\zeta}^*) < 0$, $i = 1,..., n + r + 2$. Thus $P_\delta(x; \boldsymbol{\zeta}^*)$ equioscillates at $n + r + 3$ points between c and $-c$. From Theorem 3.3, there exists the G.P.S. $P_{r+1}(x)$ with $r + 1$ knots and $n + r + 2$ points of equioscillation between 1 and -1. Since $c > 1$, $P_\delta(x; \boldsymbol{\zeta}^*) - P_{r+1}(x)$ has at least $n + r + 2$ sign changes in $[0, 1]$. As a result of Theorem 3.1, $\sigma_{r+1} = \| NP_{r+1}\|_\infty \leqslant \| NP_\delta(\cdot; \boldsymbol{\zeta}^*)\|_\infty = \sigma + \delta$. But this is a contradiction. Thus $c \leqslant 1$.

Case 3. For all $\boldsymbol{\zeta} \in Z$, $\sum_{i=1}^{n+r+3} R_i(\boldsymbol{\zeta}) > 0$.

Consider the mapping

$$\zeta'_i = R_i(\boldsymbol{\zeta})\Big/\sum_{j=1}^{n+r+3} R_j(\boldsymbol{\zeta}), \qquad i = 1,\ldots, n+r+3.$$

Since $R_i(\boldsymbol{\zeta})$ is a continuous function of $\boldsymbol{\zeta} \in Z$, there exists, by the Brouwer fixed-point theorem, a $\boldsymbol{\zeta}^* \in Z$ such that

$$\zeta^*_i = R_i(\boldsymbol{\zeta}^*)\Big/\sum_{j=1}^{n+r+3} R_j(\boldsymbol{\zeta}^*), \qquad i = 1,\ldots, n+r+3. \tag{4.2}$$

Since $M(\boldsymbol{\zeta}^*) = \max_i m_i(\boldsymbol{\zeta}^*)$, there exists an i_0 such that $R_{i_0}(\boldsymbol{\zeta}^*) = 0$ $(M(\boldsymbol{\zeta}^*) = m_{i_0}(\boldsymbol{\zeta}^*))$. From (4.2), $\zeta^*_{i_0} = 0$ and thus

$$\begin{aligned}\| P_\delta(\cdot; \boldsymbol{\zeta}^*)\|_\infty = M(\boldsymbol{\zeta}^*) = m_{i_0}(\boldsymbol{\zeta}^*) &= | P_\delta(x_{i_0}(\boldsymbol{\zeta}^*); \boldsymbol{\zeta}^*)| \\ &= | f(x_{i_0}(\boldsymbol{\zeta}^*))| \leqslant 1.\end{aligned}$$

Thus, for each $\sigma \in [\sigma_r, \sigma_{r+1})$, $f \in W_\infty^{(n)}(\mathscr{K}_\epsilon; \sigma)$ and $\delta > 0$, small, there exists a G.P.S. $P_\delta(x)$ with exactly $r + 4$ knots such that $P_\delta^{(j)}(\xi) = f^{(j)}(\xi)$, $j = k-1, k$, $\| P_\delta \|_\infty \leqslant 1$, and $\| NP_\delta \|_\infty = \sigma + \delta$. Let $\delta \downarrow 0$. Since the class of G.P.S.'s with at most $r + 4$ knots in $W_\infty^{(n)}(\mathscr{K}_\epsilon; \sigma + \delta)$ is closed and compact, it follows that there exists a G.P.S. $P(x)$ with at most $r + 4$ knots such that $\| P \|_\infty \leqslant 1$, $\| NP \|_\infty = \sigma$, and $P^{(j)}(\xi) = f^{(j)}(\xi)$, $j = k-1, k$. The theorem is proved.

Remark 4.1. By a more careful analysis, it is possible to prove that we can, in fact, reduce the admissible class of generalized perfect splines to those with at most $r + 3$ knots.

Remark 4.2. If $f \in W_\infty^{(n)}(\mathscr{K}_\epsilon; \sigma)$, then we may perturb $f(x)$ by any $g(x) = \sum_{i=1}^{n-1} a_i u_i(x; \epsilon)$ so as to increase $\| f \|_\infty$ while keeping $\| Nf \|_\infty$ unchanged. It therefore follows that $f \in W_\infty^{(n)}(\mathscr{K}_\epsilon; \sigma)$ which solves (4.1) must satisfy $\| f \|_\infty = 1$.

PROPOSITION 4.1. *The supremum in* (4.1) *is attained.*

Proof. This follows by a standard compactness argument which is obviated by Theorem 4.1.

DEFINITION 4.1. Let $P(x)$ be a G.P.S. with exactly l knots and exactly $n + l - 1$ points of equioscillation $\{x_i\}_{i=1}^{n+l-1}$, $0 \leqslant x_1 \leqslant \cdots < x_{n+l-1} < 1$. We say that $P(x)$ has *opposite orientation* if $P(x_{n+l-1})\, NP(1) < 0$. (This is equivalent to $P(x_1)\, NP(0)(-1)^n < 0$.)

Let $\mathscr{P}_\epsilon(\sigma)$ denote the class of generalized perfect splines in $W_\infty^{(n)}(\mathscr{K}_\epsilon\,;\sigma)$ which maximize (4.1) for some ξ, k, λ, and μ as prescribed.

The following theorem uses a perturbation technique wihch is a variant of that used in Karlin [10, pp. 470–472].

THEOREM 4.2. *For every $P \in \mathscr{P}_\epsilon(\sigma)$, we have $\| P \|_\infty = 1$ and $\| NP \|_\infty = \sigma$. Moreover, if $P \in \mathscr{P}_\epsilon(\sigma)$ has l knots, then P has at least $n + l - 1$ points of equioscillation, and if P has exactly $n + l - 1$ points of equioscillation, then P has opposite orientation.*

Proof. If $P \in \mathscr{P}_\epsilon(\sigma)$, then $\| P \|_\infty = 1$ by Remark 4.2. Assume that $P(x)$ has l knots and m points of equioscillation $\{x_i\}_{i=1}^m$, $0 \leqslant x_1 < \cdots < x_m \leqslant 1$, where, without loss of generality, we assume $P(x_i) = (-1)^{i+m}$, $i = 1,..., m$. Choose $y_i \in (x_i, x_{i+1})$, $i = 1,..., m - 1$, such that, in $[y_i, y_{i+1}]$, $P(x)(-1)^{i+m} < 1$, $i = 0, 1,..., m - 1$, where $y_0 = 0$, $y_m = 1$.

The idea of the proof is to construct, where the conditions of the theorem are not met, a G.P.S. $Q_\delta(x)$ with at most l knots for which $Q_\delta^{(j)}(\xi) = P^{(j)}(\xi)$, $j = k - 1, k$, and such that $\| Q_\delta \|_\infty < 1$, $\| NQ_\delta \|_\infty \leqslant \sigma$. This would, by Remark 4.2, contradict the maximizing property of $P(x)$.

Note that while we have restricted our attention to [0, 1], the generalized perfect splines are themselves well defined for any x on the real line with the retention of the appropriate results of Section 3.

We construct $Q_\delta(x)$ as follows: $Q_\delta(x)$ is the G.P.S. which satisfies

(1) $Q_\delta(y_i) = P(y_i)$, $i = 1,..., m - 1$,

(2) $Q_\delta^{(j)}(\xi) = P^{(j)}(\xi)$, $j = k - 1, k$,

(3) $Q_\delta(x_m) = P(x_m) - \delta = 1 - \delta$, for $\delta > 0$, small.

Case 1. $NP(1) > 0$ and $m \leqslant n + l - 1$.

(4) Since $n + l - m - 1 \geqslant 0$, choose $n + l - m - 1$ points in $(-1, 0)$ and let $Q_\delta(x)$ interpolate $P(x)$ at these points.

We have imposed exactly $n + l + 1$ conditions upon $Q_\delta(x)$. Thus $Q_\delta(x)$ is uniquely defined with at most l knots satisfying (1)–(4). Since $P(x)$ and $Q_\delta(x)$ both have at most l knots, but are not identical by condition (3), it follows from Theorem 3.1(a) that $P(x) - Q_\delta(x)$ has no additional zeros in $(-\infty, \infty)$. Also, $Q_\delta(x)$ is a continuous function of δ and, as $\delta \downarrow 0$, $Q_\delta(x) \to P(x)$ uniformly on [0, 1]. Therefore $\| Q_\delta \| < \| P \|_\infty = 1$, and $NQ_\delta(1) \cdot NP(1) > 0$ for $\delta > 0$, small. By assumption, $NP(1) > 0$, and by construction $Q_\delta(x) < P(x)$ for all $x \geqslant 1$. Thus $NP(1) \geqslant NQ_\delta(1) > 0$, implying $\sigma \geqslant \| NP \|_\infty \geqslant \| NQ_\delta \|_\infty$.

Case 2. $NP(1) < 0$ and $m \leqslant n + l - 2$.

Conditions (1)–(3) of Case 1 specify $m + 2$ constraints on $Q_\delta(x)$. To construct $Q_\delta(x)$ as in Case 1, we must specify $n + l + 1$ conditions. Thus we have $n + l - m - 1$ remaining "degrees of freedom." Since $n + l - m - 1 > 0$, we set $Q_\delta(2) = P(2)$. The remaining $n + l - m - 2$ conditions, if any, are are specified as in condition (4) of Case 1. As above, $Q_\delta(x) \to P(x)$ as $\delta \downarrow 0$, and $\| Q_\delta \|_\infty < \| P \|_\infty = 1$ for $\delta > 0$, small, since $P(x) - Q_\delta(x)$ has no additional zeros in $(-\infty, \infty)$. Since $P(2) = Q_\delta(2)$, $Q_\delta(x) > P(x)$ for $x > 2$. Recall that $NP(1) < 0$, and $NQ_\delta(1) \cdot NP(1) > 0$ for δ sufficiently small. The analysis now follows that of Case 1.

It remains to prove that for $P \in \mathscr{P}_\epsilon(\sigma)$, $\| NP \|_\infty = \sigma$. This fact may be proved via the analysis of Case 1 or 2. We do not enter into the details. The theorem is proved.

We now wish to return to a consideration of ordinary perfect splines, i.e., to the study of

$$\max_{f \in W_\infty^{(n)}(\sigma)} | \lambda f^{(k)}(\xi) + \mu f^{(k-1)}(\xi) | \tag{4.3}$$

for some ξ, k, μ, and λ as stipulated earlier, where $W_\infty^{(n)}(\sigma) = \{f : f \in W_\infty^{(n)}, \|f\|_\infty \leqslant 1, \| f^{(n)} \|_\infty \leqslant \sigma\}$.

If $P(x; \epsilon)$, for each $\epsilon > 0$, is a generalized perfect spline (see (3.1)) such that $\| P(\cdot; \epsilon)\|_\infty = 1$, $\| NP(\cdot; \epsilon)\|_\infty = \sigma$, and $P(x; \epsilon)$ has l knots and $m > 0$ points of equioscillation then as $\epsilon \downarrow 0$, by an appropriate choice of subsequences, we obtain in the limit an ordinary perfect spline $P(x)$ for which $\| P \|_\infty = 1$, $\| P^{(n)} \|_\infty = \sigma$, and $P(x)$ has at most l knots and at least m points of equioscillation. (For a more detailed proof of the limiting behaviour, see Karlin [8, 9]. Suffice it to say that since the convergence is uniform on [0, 1], we cannot lose points of equioscillation, nor can we gain knots.)

Theorems 4.1 and 4.2 lead to the following two results.

THEOREM 4.3. *If* $\sigma \in [\sigma_r, \sigma_{r+1})$, *then for each* $f \in W_\infty^{(n)}(\sigma)$, *there exists a perfect spline* $P \in W_\infty^{(n)}(\sigma)$ *with at most* $r + 4$ *knots and such that* $\| P^{(n)} \|_\infty = \sigma$, $P^{(j)}(\xi) = f^{(j)}(\xi)$, $j = k - 1, k$.

THEOREM 4.4. *It is sufficient in the study of* (4.3) *to consider perfect splines* $P(x)$ *for which* $\| P \|_\infty = 1$, $\| P^{(n)} \|_\infty = \sigma$ *and which further satisfy the conditions that if* $P(x)$ *has* l *knots, then* $P(x)$ *has at least* $n + l - 1$ *points of equioscillation, and if* $P(x)$ *has exactly* $n + l - 1$ *points of equioscillation, then* $P(x)$ *has opposite orientation, i.e.,* $P^{(n)}(1)\, P(x_{n+l-1}) < 0$, *where* $\{x_i\}_{i=1}^{n+l-1}$, $0 \leqslant x_1 < \cdots < x_{n+l-1} \leqslant 1$ *are the points of equioscilltaion of* $P(x)$ *on* [0, 1].

Remark 4.3. We do not claim that every function $f \in W_\infty^{(n)}(\sigma)$ maximizing (4.3) for some ξ, k, λ, and μ is necessarily a perfect spline.

5. THE MAIN THEOREM

We define, for each $\sigma \in (0, \infty)$, a class of perfect splines $\mathscr{P}(\sigma)$ such that any $P \in \mathscr{P}(\sigma)$ satisfies $\| P \|_\infty = 1$ and $\| P^{(n)} \|_\infty = \sigma$, and for which

(A) If $\sigma = \sigma_r$, $r \geqslant 0$, then for any $P \in \mathscr{P}(\sigma)$, either

(1) $P(x) = P_r(x)$, or

(2) $P(x)$ has $r + 1$ knots, $n + r$ points of equioscillation, and opposite orientation.

(Recall that $P_r(x)$ is the perfect spline, unique up to multiplication by -1, with r knots and $n + r + 1$ points of equioscillation which satisfies $\| P \|_\infty = 1$, $\| P^{(n)} \|_\infty = \sigma_r$.)

(B) If $\sigma \in (\sigma_r, \sigma_{r+1})$, $r \geqslant -1$, then for any $P \in \mathscr{P}(\sigma)$, one of the following holds:

(1) $P(x)$ is a Zolotarev perfect spline $Z(x; \sigma)$,

(2) $P(x)$ has $r + 1$ knots, $n + r$ points of equioscillation, and opposite orientation,

(3) $P(x)$ has $r + 2$ knots, $n + r + 1$ points of equioscillation, and opposite orientation.

On the basis of Theorems 4.3 and 4.4, we shall prove the following result.

THEOREM 5.1. *If $\sigma > 0$, then on considering*

$$\max_{f \in W_\infty^{(n)}(\sigma)} | \lambda f^{(k)}(\xi) + \mu f^{(k-1)}(\xi)| \tag{5.1}$$

for $\xi \in [0, 1]$, $1 \leqslant k \leqslant n - 1$, λ, μ real, all fixed, it is sufficient to consider the class of perfect splines $\mathscr{P}(\sigma)$.

The proof of Theorem 5.1 readily follows from the following proposition.

PROPOSITION 5.1. *If $P(x)$ is a perfect spline satisfying $\| P \|_\infty = 1$, with exactly $r + 1$ knots, $n + r$ points of equioscillation, and opposite orientation, then*

$$\sigma_{r-1} < \| P^{(n)} \|_\infty < \sigma_{r+1}.$$

An important tool in the proof of Proposition 5.1 and in the subsequent section is a simple version of the Budan–Fourier theorem for splines. For the statement of the theorem, we need the following definitions.

DEFINITION 5.1. Let $x = (x_1, \ldots, x_l)$ be a real vector of l components. Then

(i) $S^-(x)$ denotes the number of actual sign changes in the sequence $x_1, \ldots, x_l$ with zero terms omitted.

(ii) $S^+(x)$ counts the maximum number of sign changes in the sequence $x_1, \ldots, x_l$, where zero terms are assigned values $+1$ and -1, arbitrarily.

For example,

$$S^-(-1, 0, 1, -1, 0, -1) = 2, \qquad S^+(-1, 0, 1, -1, 0, -1) = 4.$$

DEFINITION 5.2. If $f \in C[a, b]$, $f(x) \not\equiv 0$ in any subinterval (c, d), of $[a, b]$, then $\tilde{Z}_f(a, b)$ counts the number of zeros of $f(x)$ in (a, b) with the convention that if $f(x) = 0$, but f does not change sign at x, then the zero of f at x is counted twice.

DEFINITION 5.3. A step function $g(x)$ on (a, b) with a finite number of jumps can always be written in the form

$$g(x) = \sum_{i=0}^{l} a_i (x - \xi_i)_+^0, \qquad \text{where} \qquad \xi_0 = a < \xi_1 < \cdots < \xi_l < b.$$

For such a function, we define

$$S^-_{(a,b)}(g) = S^-\left(a_0, a_0 + a_1, \ldots, \sum_{i=0}^{l} a_i\right),$$

i.e., $S^-_{(a,b)}(g)$ counts the number of strict sign changes of $g(x)$ on (a, b).

On the basis of the above definitions, we may now state a version of the Budan–Fourier theorem for splines.

THEOREM 5.2 (de Boor and Schoenberg [3], Melkman [15]). *If $s(x)$ is a polynomial spline function of exact degree n on $[a, b]$ (i.e., $s^{(n)}(x) \neq 0$ for some $x \in (a, b)$), with finitely many (active) knots in (a, b), all simple, and if $s(x) \neq 0$ on any subinterval of (a, b), then,*

$$\tilde{Z}_s(a, b) \leqslant S^-_{(a,b)}(s^{(n)}) + S^-(s(a), s'(a), \ldots, s^{(n)}(a+)) \\ - S^+(s(b), s'(b), \ldots, s^{(n)}(b-)).$$

The following lemma is used in the proof of Proposition 5.1 and in Section 6.

LEMMA 5.1. *Assume $P(x)$ is a perfect spline with $r+1$ knots $\{\eta_i\}_1^{r+1}$, $0<\eta_1<\cdots<\eta_{r+1}<1$, $n+r$ points of equioscillation $\{x_i\}_{i=1}^{n+r}$, $0\leqslant x_1<\cdots<x_{n+r}\leqslant 1$, and opposite orientation. Then*

$$x_i<\eta_i<x_{i+n-1}\,,\qquad i=1,\ldots,r+1. \tag{5.2}$$

Proof. $P'(x)$ vanishes at $x_2,\ldots,x_{n+r-1}$. Furthermore, if $x_1>0$, then $P'(x_1)=0$, while if $x_1=0$, then since $P(x)$ has opposite orientation, there exists a $y_1\leqslant 0$ such that $P'(y_1)=0$. Similarly, if $x_{n+r}<1$, $P'(x_{n+r})=0$, while if $x_{n+r}=1$, there exists a $y_{n+r}\geqslant 1$ such that $P'(y_{n+r})=0$. From Proposition 2.2 applied to the perfect spline $P'(x)$ of degree $n-1$, and since $0<\eta_1,\eta_{r+1}<1$, it follows that

$$x_i<\eta_i<x_{i+n-1}\,,\qquad i=1,\ldots,r+1.$$

Proof of Proposition 5.1. The proof is divided into four cases.

Case 1

Assume $\|P^{(n)}\|_\infty=\sigma<\sigma_{r-1}=\|P_{r-1}^{(n)}\|_\infty$.

Both $P(x)$ and $P_{r-1}(x)$ exhibit $n+r$ points of equioscillation on [0, 1]. Let $\{x_i\}_{i=1}^{n+r}$ and $\{y_i\}_{i=1}^{n+r}$, $0\leqslant x_1<\cdots<x_{n+r}\leqslant 1$, $0=y_1<\cdots<y_{n+r}=1$, denote the points of equioscillation of $P(x)$ and $P_{r-1}(x)$ on [0, 1], respectively. Assume, without loss of generality, that $P(x_i)=P_{r-1}(y_i)$, $i=1,\ldots,n+r$. Since $\sigma<\sigma_{r-1}$, $P(x)-P_{r-1}(x)$ cannot vanish identically on any subinterval of [0, 1], and therefore $P(x)-P_{r-1}(x)$ has at least $n+r$ zeros on [0, 1]. Thus $P^{(n)}(x)-P_{r-1}^{(n)}(x)$ has at least r sign changes on [0, 1]. But $P_{r-1}(x)$ has $r-1$ knots and $\sigma_{r-1}>\sigma$. Therefore $P^{(n)}(x)-P_{r-1}^{(n)}(x)$ has at most $r-1$ sign changes on [0, 1], a contradiction.

Case 2

Assume $\|P^{(n)}\|_\infty=\sigma>\sigma_{r+1}=\|P_{r+1}^{(n)}\|_\infty$.

Choose $\epsilon>0$, sufficiently small, such that $(1-\epsilon)\sigma>\sigma_{r+1}$. Thus $(1-\epsilon)P^{(n)}(x)\pm P_{r+1}^{(n)}(x)$ has exactly $r+1$ sign changes on [0, 1], whose orientation is totally determined by $P^{(n)}(x)$. Since $P_{r+1}(x)$ has $n+r+2$ points of equioscillation and $\|(1-\epsilon)P\|_\infty=1-\epsilon<\|P_{r+1}\|_\infty$, $(1-\epsilon)P(x)\pm P_{r+1}(x)$ has at least $n+r+1$ sign changes in (0, 1), and thus $(1-\epsilon)P^{(n)}(x)\pm P_{r+1}^{(n)}(x)$ has at least $r+1$ sign changes on (0, 1). By the previous remarks, $(1-\epsilon)P^{(n)}(x)\pm P_{r+1}^{(n)}(x)$ has exactly $r+1$ sign changes. However, the orientation of the sign changes in this second method is determined by $\pm P_{r+1}(x)$. A contradiction.

Case 3

Assume $\|P^{(n)}\|_\infty=\sigma_{r-1}=\|P_{r-1}^{(n)}\|_\infty$.

Since we must consider the possibility of $P(x)-P_{r-1}(x)$ vanishing on some subinterval of [0, 1], the analysis is slightly more complicated than that of Cases 1 and 2. Let $\{\eta_i\}_{i=1}^{r+1}$, $0 < \eta_1 < \cdots < \eta_{r+1} < 1$, and $\{x_i\}_{i=1}^{n+r}$, $0 \leqslant x_1 < \cdots < x_{n+r} \leqslant 1$, denote the knots and points of equioscillation of $P(x)$, respectively, and let $\{\xi_i\}_{i=1}^{r-1}$, $0 < \xi_1 < \cdots < \xi_{r-1} < 1$, and $\{y_i\}_{i=1}^{n+r}$, $0 = y_1 < y_2 < \cdots < y_{n+r} = 1$ denote the knots and points of equioscillation of $P_{r-1}(x)$, respectively. Since $P'_{r-1}(y_i) = 0$, $i = 2,..., n + r - 1$, it follows from Proposition 2.2 that $y_{i+1} < \xi_i < y_{i+n}$, $i = 1,..., r - 1$. Assume, without loss of generality, that $P(x_{n+r}) = P_{r-1}(y_{n+r})$.

Since $P(x)$ has opposite orientation, $P_{r-1}^{(n)}(0)\, P^{(n)}(0) < 0$ and $P_{r-1}^{(n)}(1)\, P^{(n)}(1) < 0$, and therefore $P(x) - P_{r-1}(x)$ cannot vanish identically on $[0, \epsilon]$ or on $[1 - \epsilon, 1]$ for $\epsilon > 0$, small. We wish to prove that $P(x) - P_{r-1}(x)$ never vanishes identically on any subinterval of [0, 1]. Assume the converse.

Subcase 3.1. There exists a ξ_i such that $P(x) - P_{r-1}(x)$ does not vanish identically on any subinterval of $[0, \xi_i]$, but $P(x) - P_{r-1}(x) \equiv 0$ on $(\xi_i, \xi_i + \epsilon)$, $\epsilon > 0$, small.

Since $y_{i+1} < \xi_i$, $P(x) - P_{r-1}(x)$ has at least i zeros in $[0, \xi_i)$, and thus at least $i - 1$ in $(0, \xi_i)$. (We are counting zeros as in the definition of $\tilde{Z}$.) Furthermore, $S^-_{(0,\xi_i)}(P^{(n)} - P_{r-1}^{(n)}) \leqslant i - 1$, and $P^{(l)}(\xi_i) - P_{r-1}^{(l)}(\xi_i) = 0$, $l = 0, 1,..., n - 1$. We now apply Theorem 5.2 to obtain a contradiction.

Subcase 3.2. There exists an η_i such that $P(x) - P_{r-1}(x)$ does not vanish identically on any subinterval of $[0, \eta_i]$, but $P(x) - P_{r-1}(x) \equiv 0$ on $(\eta_i, \eta_i + \epsilon)$, $\epsilon > 0$, small, and $\eta_i \neq \xi_i$, $j = 1,..., r - 1$.

Thus $\xi_j < \eta_i < \xi_{j+1}$ for some $j = 0, 1,..., r - 1$, where $\xi_0 = 0$, $\xi_r = 1$. If $j = 0$ or $i = 1$, then $P^{(n)}(x) - P_{r-1}^{(n)}(x)$ has no sign change on $(0, \eta_i)$. However, $y_1 = 0 \leqslant x_1 < \eta_1 \leqslant \eta_i$ (see (5.2)). Thus $P(x) - P_{r-1}(x)$ has at least one zero in $[0, \eta_i)$. A contradiction immediately follows from Theorem 5.2. Thus $\xi_j < \eta_i < \xi_{j+1}$ for some $j = 1,..., r - 1$, and $i > 1$. Now, $S^-_{(0,\eta_i)}(P^{(n)} - P_{r-1}^{(n)}) \leqslant \min\{j, i - 1\}$, and since $x_i < \eta_i$ from (5.2), $P(x) - P_{r-1}(x)$ has at least $i - 1$ zeros in $[0, \eta_i)$, and at least $i - 2$ in $(0, \eta_i)$. An application of Theorem 5.2 now yields $i - 1 \leqslant \min\{j, i - 1\}$. Therefore $j \geqslant i - 1$, and $y_i < \xi_{i-1} \leqslant \xi_j < \eta_i$. However, since y_i, $x_i < \eta_i$, $P(x) - P_{r-1}(x)$ has at least i zeros in $[0, \eta_i)$ (and $i - 1$ zeros in $(0, \eta_i)$). A contradiction ensues.

Since $P(x) - P_{r-1}(x)$ cannot vanish identically in any subinterval of [0, 1], and $P(x)$ and $P_{r-1}(x)$ both equioscillate at $n + r$ points with the same orientation, (i.e., $P(x_i) = P_{r-1}(y_i)$, $i = 1,..., n + r$), the analysis of Case 1 is applicable. Case 3 is proved.

Case 4

Assume $\| P^{(n)} \|_\infty = \sigma_{r+1} = \| P_{r+1}^{(n)} \|_\infty$.

Orient $P(x)$ and $P_{r+1}(x)$ so that $P^{(n)}(1) = P_{r+1}^{(n)}(1)$. Since $P(x)$ and $P_{r+1}(x)$

both have $r+1$ knots, $S^-_{(0,1)}(P^{(n)} - P^{(n)}_{r+1}) \leqslant r$. Furthermore, $P_{r+1}(1) \neq P(1)$, and $P_{r+1}(0) \neq P(0)$ due to the opposite orientation of $P(x)$. Thus $P(x) - P_{r+1}(x)$ cannot vanish identically in $[0, \epsilon]$ or $[1-\epsilon, 1]$, for $\epsilon > 0$, small. If $P(x) - P_{r+1}(x)$ does not vanish identically on any subinterval of $[0, 1]$, then since $P_{r+1}(x)$ has $n+r+2$ points of equioscillation, $P(x) - P_{r+1}(x)$ has at least $n+r+1$ zeros on $[0, 1]$, hence on $(0, 1)$, and application of Theorem 5.2 immediately leads to a contradiction. As above, we prove that the hypothesis that $P(x) - P_{r+1}(x)$ vanishes on some subinterval of $[0, 1]$ is untenable.

We assume, as previously, that $\{\eta_i\}_{i=1}^{r+1}$ and $\{x_i\}_{i=1}^{n+r}$ are the knots and points of equioscillation of $P(x)$, respectively. Let $\{\xi_i\}_{i=1}^{r+1}$ and $\{y_i\}_{i=1}^{n+r+2}$ denote the ordered set of knots and points of equioscillation, respectively, of $P_{r+1}(x)$.

Subcase 4.1. There exists a ξ_i such that $P(x) - P_{r+1}(x)$ does not vanish identically on any subinterval of $[0, \xi_i]$, but $P(x) - P_{r+1}(x) = 0$ on $(\xi_i, \xi_i + \epsilon)$, $\epsilon > 0$, small.

A contradiction follows as in Subcase 3.1.

Subcase 4.2. There exists an η_i such that $P(x) - P_{r+1}(x)$ does not vanish identically on any subinterval of $[0, \eta_i]$, but $P(x) - P_{r+1}(x) = 0$ on $(\eta_i, \eta_i + \epsilon)$, $\epsilon > 0$, small, and $\eta_i \neq \xi_j$, $j = 1,..., r+1$.

Thus $\xi_j < \eta_i < \xi_{j+1}$ for some $j = 0, 1,..., r+1$, where $\xi_0 = 0$, $\xi_{r+2} = 1$. If $j = 0$, then $0 < \eta_i < \xi_1$, and since $P^{(n)}(0) = P^{(n)}_{r+1}(0)$, it is necessary that $i \geqslant 2$ (i be even) in order that $P(x) - P_{r+1}(x) \equiv 0$ on $(\eta_i, \eta_i + \epsilon)$. However, $0 \leqslant x_1 < x_2 < \eta_2 \leqslant \eta_i$ implying that $P(x) - P_{r+1}(x)$ has at least one zero in $[0, \eta_i)$, while $S^-_{(0,\eta_i)}(P^{(n)} - P^{(n)}_{r+1}) = 0$. A contradiction follows from Theorem 5.2. For $j > 0$, $S^-_{(0,\eta_i)}(P^{(n)} - P^{(n)}_{r+1}) \leqslant \min\{j, i-1\}$, unless $j = i-1$, in which case the bound is $j - 1 = i - 2$. Since $x_i < \eta_i$ and $y_{j+1} < \xi_j < \eta_i$, $P(x) - P_{r+1}(x)$ has at least $\max\{j, i-1\}$ zeros in $[0, \eta_i)$. An application of Theorem 5.2 implies $j = i - 1$. A contradiction now follows from the above remarks.

The proposition is proved.

Proof of Theorem 5.1. On the basis of Theorems 4.3 and 4.4, it is sufficient to consider perfect splines with a finite number of knots, which are of norm 1, and whose nth derivative is of norm σ. Certainly $Z(x; \sigma)$, the Zolotarev perfect spline, satisfies the condition of Theorem 4.4 (if $\sigma = \sigma_r$, then $Z(x; \sigma_r) \equiv P_r(x)$). If $P(x)$ is any perfect spline with l knots and at least $n + l$ points of equioscillation, then $P(x) = Z(x; \gamma)$ for some γ. However, $\| Z^{(n)}(x; \gamma)\|_\infty = \gamma$, so that if $P(x) \neq Z(x; \sigma)$, $P(x)$ has l knots, $n + l - 1$ points of equioscillation, and opposite orientation. From Proposition 5.1, $\sigma_{l-2} < \| P^{(n)} \|_\infty < \sigma_l$. The theorem now follows.

Q.E.D.

6. Numerical Differentiation Formulas

In Section 5, we proved that in the study of

$$\max_{f \in W_\infty^{(n)}(\sigma)} |\lambda f^{(k)}(\xi) + \mu f^{(k-1)}(\xi)|, \tag{6.1}$$

where $1 \leqslant k \leqslant n-1$, $\xi \in [0, 1]$, and λ, μ, real, all fixed, it is sufficient to restrict ourselves to a consideration of the class of perfect splines $\mathscr{P}(\sigma)$.

In this section, we study in greater detail a special case of (6.1), namely,

$$\max_{f \in W_\infty^{(n)}(\sigma)} |f^{(k)}(\xi)|, \tag{6.2}$$

where $1 \leqslant k \leqslant n-1$ and $\xi \in [0, 1]$, fixed.

From this study it will follow that for each k, $1 \leqslant k \leqslant n-1$, and each $P \in \mathscr{P}(\sigma)$, there exists a least one point $\xi \in [0, 1]$ such that P maximizes (6.2).

A numerical differentiation formula is any equation of the form

$$f^{(k)}(\xi) = \sum_{i=1}^{p} a_i f(y_i) + \int_0^1 K(t) f^{(n)}(t)\, dt, \tag{6.3}$$

which is valid for all $f \in W_\infty^{(n)}$. Given the formula (6.3), it then follows that

$$|f^{(k)}(\xi)| \leqslant \|f\|_\infty \sum_{i=1}^{p} |a_i| + \|f^{(n)}\|_\infty \int_0^1 |K(t)|\, dt. \tag{6.4}$$

Given $P \in \mathscr{P}(\sigma)$ and k, $1 \leqslant k \leqslant n-1$, we shall find $\xi \in [0, 1]$, $\{a_i\}_{i=1}^{p}$, and $K(t)$ satisfying (6.3) for which equality holds in (6.4) for P, implying that P maximizes (6.2) for k and ξ, as above. (For a discussion of numerical differentiation formulas on $[0, 1]$, see Kallioniemi [6].)

Proposition 6.1. *Let $P(x)$ be a perfect spline with $r+1$ knots, $\{\eta_i\}_{i=1}^{r+1}$, $0 < \eta_1 < \cdots < \eta_{r+1} < 1$, exactly $n+r$ points of equioscillation $\{x_i\}_{i=1}^{n+r}$, $0 \leqslant x_1 < \cdots < x_{n+r} \leqslant 1$, and opposite orientation. Then there exists a nontrivial spline $s(x)$, unique up to multiplication by a constant, of degree $n-1$ with the $r+1$ knots $\{\eta_i\}_{i=1}^{r+1}$ which vanishes at the $\{x_i\}_{i=1}^{n+r}$, i.e.,*

$$s(x) = \sum_{i=0}^{n-1} b_i x^i + \sum_{i=1}^{r+1} c_i (x - \eta_i)_+^{n-1},$$

and $s(x_i) = 0$, $i = 1,\ldots, n+r$, $s(x) \not\equiv 0$.

Furthermore, $s^{(l)}(x)$ is not identically zero on any subinterval (a, b) of $[0, 1]$, $l = 0, 1,\ldots, n-1$.

Proof. From (5.2) we have $x_i < \eta_i < x_{i+n-1}$, $i = 1,\ldots, r+1$. These are explicitly the conditions (see Schoenberg and Whitney [18]) necessary to insure the existence of the nontrivial $s(x)$ as in the statement of the proposition, uniquely determined up to a multiplicative constant, and such that $s(x) = 0$ iff $x = x_i$, $i = 1,\ldots, n+r$. Thus, $s(x)$ changes sign at the $\{x_i\}_{i=1}^{n+r}$, and also, $s^{(n-1)}(x) \not\equiv 0$ on $[0, 1]$. From Theorem 5.2,

$$\begin{aligned} n + r &\leqslant S^-_{(0,1)}(s^{(n-1)}) + S^-(s(0),\ldots, s^{(n-2)}(0), s^{(n-1)}(0)) \\ &\quad - S^+(s(1),\ldots, s^{(n-2)}(1), s^{(n-1)}(1)) \\ &\leqslant r + 1 + n - 1 = n + r. \end{aligned}$$

Therefore equality holds throughout, implying that $s(x)$ is a polynomial of exact degree $n-1$ in each interval (η_{i-1}, η_i), $i = 1,\ldots, r+2$, where $\eta_0 = 0$, $\eta_{r+2} = 1$, and hence $s^{(l)}(x)$ does not vanish identically on any subinterval (a, b) of $[0, 1]$, $l = 0, 1,\ldots, n-1$. Q.E.D.

By Proposition 6.1, $s^{(k)}(x)$ has exactly $n + r - k$ simple zeros and vanishes nowhere else on $[0, 1]$, $k = 1,\ldots, n-2$, while $s^{(n-1)}(x)$ has $r+1$ sign changes at $\{\eta_i\}_{i=1}^{r+1}$. Let $L_k(\xi)$ denote the matrix

$$\begin{bmatrix} 1 & \cdots & 1 & 0 \\ x_1 & \cdots & x_{n+r} & 0 \\ \vdots & & \vdots & \vdots \\ x_1^{k-1} & \cdots & x_{n+r}^{k-1} & 0 \\ x_1^{k} & \cdots & x_{n+r}^{k} & k! \\ x_1^{k+1} & \cdots & x_{n+r}^{k+1} & \dfrac{(k+1)!}{1!}\,\xi \\ \vdots & & \vdots & \vdots \\ x_1^{n-1} & \cdots & x_{n+r}^{n-1} & \dfrac{(n-1)!}{(n-1-k)!}\,\xi^{n-1-k} \\ (x_1-\eta_1)_+^{n-1} & \cdots & (x_{n+r}-\eta_1)_+^{n-1} & \dfrac{(n-1)!}{(n-1-k)!}\,(\xi-\eta_1)_+^{n-1-k} \\ \vdots & & \vdots & \vdots \\ (x_1-\eta_{r+1})_+^{n-1} & \cdots & (x_{n+r}-\eta_{r+1})_+^{n-1} & \dfrac{(n-1)!}{(n-1-k)!}\,(\xi-\eta_{r+1})_+^{n-1-k} \end{bmatrix}.$$

Since

$$\begin{aligned} (b_0,\ldots, b_{n-1}, c_1,\ldots, c_{r+1})\, L_k(\xi) &= (s(x_1),\ldots, s(x_{n+r}), s^{(k)}(\xi)) \\ &= (0,\ldots, 0, s^{(k)}(\xi)), \end{aligned}$$

it follows that $\det L_k(\xi) = 0$ at the points ξ for which $s^{(k)}(\xi) = 0$. Thus, for $k = 1, 2, \ldots, n-2$, there exist at least $n + r - k$ points $\{\xi_i^{(k)}\}_{i=1}^{n+r-k}$ for which $\det L_k(\xi_i^{(k)}) = 0$, $i = 1, \ldots, n + r - k$. (It may, in fact, be shown that $L_k(\xi)$ changes sign at $\{\xi_i^{(k)}\}_{i=1}^{n+r-k}$ and vanishes nowhere else in $[0, 1]$.)

Since any $f \in W_\infty^{(n)}$ may be written in the form

$$f(x) = \sum_{i=0}^{n-1} d_i x^i + \frac{1}{(n-1)!} \int_0^1 (x - t)_+^{n-1} f^{(n)}(t)\, dt,$$

the existence of a numerical quadrature formula of the form

$$f^{(k)}(\xi) = \sum_{i=1}^{p} a_i f(y_i) + \frac{1}{(n-1)!} \int_0^1 K(t) f^{(n)}(t)\, dt$$

(see (6.3)) which is valid for all $f \in W_\infty^{(n)}$, where $1 \leqslant k \leqslant n-1$, $\xi \in [0, 1]$, $0 \leqslant y_1 < \cdots < y_p \leqslant 1$, is equivalent to the existence of numbers $\{a_i\}_1^p$, such that

$$\begin{aligned} &\sum_{i=1}^{p} a_i y_i^{\,l} = 0, && l = 0, 1, \ldots, k-1, \\ &\sum_{i=1}^{p} a_i y_i^{\,l} = \frac{l!}{(l-k)!}\, \xi^{l-k}, && l = k, \ldots, n-1, \end{aligned} \tag{6.5}$$

and

$$K(t) = K_\xi(t) = \left[\frac{(n-1)!}{(n-1-k)!} (\xi - t)_+^{n-1-k} - \sum_{i=1}^{p} a_i (y_i - t)_+^{n-1}\right]. \tag{6.6}$$

Let $P(x)$ be the perfect spline considered in Proposition 6.1.

PROPOSITION 6.2. *For each $\xi \in [0, 1]$, there exists a numerical differentiation formula of the form*

$$f^{(k)}(\xi) = \sum_{i=1}^{n+r} a_i(\xi) f(x_i) + \frac{1}{(n-1)!} \int_0^1 K_\xi(t) f^{(n)}(t)\, dt \tag{6.7}$$

such that $K_\xi(\eta_i) = 0$, $i = 1, \ldots, r$, where $\{x_i\}_{i=1}^{n+r}$ and $\{\eta_i\}_{i=1}^{r}$ are as in Proposition 6.1.

Proof. The proof of Proposition 6.2 is equivalent to solving the linear system

$$\begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_{n+r} \\ \vdots & & \vdots \\ x_1^{k-1} & \cdots & x_{n+r}^{k-1} \\ x_1^{k} & \cdots & x_{n+r}^{k} \\ x_1^{k+1} & \cdots & x_{n+r}^{k+1} \\ \vdots & & \vdots \\ x_1^{n-1} & \cdots & x_{n+r}^{n-1} \\ (x_1-\eta_1)_+^{n-1} & \cdots & (x_{n+r}-\eta_1)_+^{n-1} \\ \vdots & & \vdots \\ (x_1-\eta_r)_+^{n-1} & \cdots & (x_{n+r}-\eta_r)_+^{n-1} \end{bmatrix} \begin{bmatrix} a_1(\xi) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{n+r}(\xi) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ k! \\ \dfrac{(k+1)!}{1!}\xi \\ \vdots \\ \dfrac{(n-1)!}{(n-1-k)!}\xi^{n-1-k} \\ \dfrac{(n-1)!}{(n-1-k)!}(\xi-\eta_1)_+^{n-1-k} \\ \vdots \\ \dfrac{(n-1)!}{(n-1-k)!}(\xi-\eta_r)_+^{n-1-k} \end{bmatrix} \tag{6.8}$$

This can be achieved since the nonsingularity of the above matrix is equivalent to $x_i < \eta_i < x_{i+n}$, $i = 1,\ldots, r$. For $P(x)$ as above, $x_i < \eta_i < x_{i+n-1} < x_{i+n}$, $i = 1,\ldots, r$, and the result follows. Q.E.D.

For certain choices of ξ, we can make K_ξ vanish at η_{r+1}, too. We have

$$K_\xi(\eta) = \det \begin{bmatrix} 1 & \cdots & 1 & 0 \\ x_1 & \cdots & x_{n+r} & 0 \\ \vdots & & \vdots & \vdots \\ x_1^{k} & \cdots & x_{n+r}^{k} & k! \\ \vdots & & \vdots & \vdots \\ x_1^{n-1} & \cdots & x_{n+r}^{n-1} & \dfrac{(n-1)!}{(n-1-k)!}\xi^{n-1-k} \\ (x_1-\eta_1)_+^{n-1} & \cdots & (x_{n+r}-\eta_1)_+^{n-1} & \dfrac{(n-1)!}{(n-1-k)!}(\xi-\eta_1)_+^{n-1-k} \\ \vdots & & \vdots & \vdots \\ (x_1-\eta_r)_+^{n-1} & \cdots & (x_{n+r}-\eta_r)_+^{n-1} & \dfrac{(n-1)!}{(n-1-k)!}(\xi-\eta_r)_+^{n-1-k} \\ (x_1-\eta)_+^{n-1} & \cdots & (x_{n+r}-\eta)_+^{n-1} & \dfrac{(n-1)!}{(n-1-k)!}(\xi-\eta)_+^{n-1-k} \end{bmatrix} D, \tag{6.9}$$

where D is the reciprocal of the determinant of the matrix given in (6.8). Therefore, $K_\xi(\eta_{r+1}) = D \det L_k(\xi)$ and, for $1 \leqslant k \leqslant n-2$, there exists $\{\xi_i^{(k)}\}_{i=1}^{n+r-k}$ such that $\det L_k(\xi_i^{(k)}) = 0$, $i = 1,..., n+r-k$. Set $\xi^* = \xi_i^{(k)}$ for some $i = 1,..., n+r-k$, and let $a_i(\xi^*) = a_i^*$, $i = 1,..., n+r$, $K_{\xi^*}(t) =: K(t)$. Thus, $K(\eta_i) = 0$, $i = 1,..., r+1$.

PROPOSITION 6.3. *For* $\{a_i^*\}$, $K(t)$ *as above,*

(1) $a_i^*(-1)^i \gamma \leqslant 0$, $i = 1,..., n+r$, *where* $\gamma = +1$ *or* -1 *fixed, and*

(2) $K(t)(-1)^i \delta - 0$ *for* $\eta_{i-1} < t < \eta_i$, $i = 1,..., r+2$, *where* $\eta_0 = 0$, $\eta_{r+2} = 1$, *and* $\delta = +1$ *or* -1, *fixed.*

Proof. Due to the possibility of degeneracies arising, various cases need be considered. Let us first note that from (6.9) and since $x_1 < \xi^* < x_{n+r}$, the support of $K(t)$ is necessarily contained in (x_1, x_{n+r}), i.e., $K^{(i)}(x_1) = K^{(i)}(x_{n+r}) = 0$, $i = 0, 1,..., n-2$, while $K^{(n-1)}(x_1-) = K^{(n-1)}(x_{n+r}+) = 0$.

Case 1

$K(t) \not\equiv 0$ on any subinterval of (x_1, x_{n+r}).

Subcase 1.1. $\xi^* \neq x_j$, $j = 2,..., n+r-1$.
By the Budan–Fourier theorem (Theorem 5.2),

$$\tilde{Z}_K(x_1, x_{n+r}) \leqslant S^-\left(\sum_{i=2}^{n+r} a_i^*, \sum_{i=3}^{n+r} a_i^*, ..., a_{n+r}^*\right) + S^-(\{K^{(i)}(\xi^*+)\}_{i=0}^{n-1}) - S^+(\{K^{(i)}(\xi^*-)\}_{i=0}^{n-1}),$$

because $S^-(\{K^{(i)}(x_1+)\}_{i=0}^{n-1}) = 0$, and $S^+(\{K^{(i)}(x_{n+r}-)\}_{i=0}^{n-1}) = n-1$. Since $\xi^* \neq x_j$, $j = 2,..., n+r$, $x_1 < \xi^* < x_{n+r}$, and $K^{(i)}(\xi^*+) = K^{(i)}(\xi^*-)$, $i = 0, 1,..., n-1$, $i \neq n-1-k$,

$$S^-(\{K^{(i)}(\xi^*+)\}_{i=0}^{n-1}) - S^+(\{K^{(i)}(\xi^*-)\}_{i=0}^{n-1}) \leqslant 2$$

for $1 \leqslant k \leqslant n-2$. Thus,

$$r+1 \leqslant \tilde{Z}_K(x_1, x_{n+r+1}) \leqslant S^-\left(\sum_{i=2}^{n+r} a_i^*, \sum_{i=3}^{n+r} a_i^*,..., a_{n+r}\right) - n + 3$$
$$\leqslant n + r - 2 - n + 3 = r + 1.$$

Therefore equality holds throughout implying that $K(t)$ changes sign at η_i, $i = 1,..., r+1$ and vanishes nowhere else in (x_1, x_{n+r}), and $S^-(\sum_{i=2}^{n+r} a_i^*, ..., a_{n+r}^*) = n+r-2$. This latter fact, together with $\sum_{i=1}^{n+r} a_i^* = 0$, implies $a_i^* a_{i+1}^* < 0$, $i = 1,..., n+r-1$.

Subcase 1.2. $\xi^* = x_j$, for some $j = 2,..., n+r-1$.

Note that if $k = 1$, then $\xi^* \neq x_j$ for any $j = 2,\ldots, n + r + 1$ from the construction of $s(x)$ in Proposition 6.1. If $1 < k \leqslant n - 2$ and $\xi^* = x_j$, then once again applying Theorem 5.2, it follows that $K(t)$ has no additional zeros other than its sign changes at $\{\eta_i\}_{i=1}^{r+1}$, and

$$S^-\left(\sum_{i=2}^{n+r} a_i^* ,\ldots, \sum_{i=j}^{n+r} a_i^*\right) = j - 2,$$

$$S^-\left(\sum_{i=j+1}^{n+r} a_i^* ,\ldots, a_{n+r}^*\right) = n + r - j - 1,$$

while $S^-(\{K^{(i)}(x_j+)\}_{i=0}^{n-1}) - S^+(\{K^{(i)}(x_j-)\}_{i=0}^{n-1}) = 3$. The latter equality implies that $(\sum_{i=j}^{n+r} a_i^*)(\sum_{i=j+1}^{n+r} a_i^*) < 0$. It follows that $a_i^* a_{i+1}^* < 0$, $i = 1,\ldots, n + r - 1$.

Case 2

$K(t) \equiv 0$ on some subinterval of (x_1, x_{n+r}).

(a) Assume that there exist i_1, i_2, $1 \leqslant i_1 < i_2 \leqslant n + r$, such that $K(t) \not\equiv 0$ on any subinterval of (x_{i_1}, x_{i_2}), and $K(t) \equiv 0$ for $t \in (x_{i_1} - \epsilon, x_{i_1})$ and $t \in (x_{i_2}, x_{i_2} + \epsilon)$ for some $\epsilon > 0$, ϵ sufficiently small.

Subcase 2.1. $\xi^* \notin [x_{i_1}, x_{i_2}]$.
By Theorem 5.2, $\tilde{Z}_K(x_{i_1}, x_{i_2}) \leqslant (i_2 - i_1 - 1) - (n - 1) = i_2 - i_1 - n$. Moreover, $x_{i_1} < \eta_{i_1}$ and $\eta_{i_2-n+1} < x_{i_2}$, by (5.2). Thus, $\tilde{Z}_K(x_{i_1}, x_{i_2}) \geqslant i_2 - i_1 - n + 2$, a contradiction.

Subcase 2.2. $\xi^* = x_{i_1}$ or $\xi^* = x_{i_2}$.
We follow the previous analysis to obtain $\tilde{Z}_K(x_{i_1}, x_{i_2}) \leqslant i_2 - i_1 - n + 1$. But as above, $\tilde{Z}_K(x_{i_1}, x_{i_2}) \geqslant i_2 - i_1 - n + 2$. A contradiction.

(b) Assume that $K(t)$ vanishes identically on (x_i, ξ^*), $\xi^* \neq x_j$, $j = 1,\ldots, n + r$, while $K(t) \not\equiv 0$ on any subinterval of (ξ^*, x_{i_2}). Adapting the previous analysis, one is again led to a contradiciton.

Thus the support of $K(t)$ is necessarily a connected interval (x_{i_1}, x_{i_2}), and $\xi^* \in (x_{i_1}, x_{i_2})$. We now reapply the analysis of Case 1, where we make use of the result (5.2), namely, $x_{i_1} < \eta_{i_1}$ and $\eta_{i_2-n+1} < x_{i_2}$. Note that for $i_2 < n + r$, $a_j^* = 0$, $j = i_2 + 1,\ldots, n + r$, while if $i_1 > 1$, then $a_j^* = 0$, $j = 1,\ldots, i_1 - 1$, and $\sum_{j=i_1}^{n+r} a_j^* = 0$.
It follows that $a_j^* a_{j+1}^* < 0$, $j = i_1,\ldots, i_2 - 1$, and $K(t)$ changes sign in (x_{i_1}, x_{i_2}) at $\eta_{i_1},\ldots, \eta_{i_2-n+1}$ and vanihses nowhere else in (x_{i_1}, x_{i_2}), implying as well that

$$\eta_{i_1-1} \leqslant x_{i_1} \quad \text{and} \quad \eta_{i_2-n+2} \geqslant x_{i_2}. \tag{6.10}$$

This proves Proposition 6.3.

THEOREM 6.1. *Let $P(x)$ be a perfect spline with $r + 1$ knots, $n + r$ points of equioscillation, and opposite orientation. Let $\xi^* = \xi_i^{(k)}$ for some $i = 1,..., n + r - k$, and $1 \leqslant k \leqslant n - 2$, as in Proposition 6.3. Then for any $f \in W_\infty^{(n)}$, $\|f\|_\infty \leqslant \| P \|_\infty$, and $\| f^{(n)} \|_\infty \leqslant \| P^{(n)} \|_\infty$, we have $| f^{(k)}(\xi^*)| \leqslant | P^{(k)}(\xi^*)|$.*

Proof.

$$f^{(k)}(\xi^*) = \sum_{i=1}^{n+r} a_i^* f(x_i) + \frac{1}{(n-1)!} \int_0^1 K_{\xi^*}(t) f^{(n)}(t)\, dt,$$

where $\{a_i^*\}_1^{n+r}$ and $K_{\xi^*}(t)$ satisfy the conditions of Proposition 6.3. Thus,

$$| f^{(k)}(\xi^*)| \leqslant \|f\|_\infty \sum_{i=1}^{n+r} | a_i^* |$$

$$+ \frac{1}{(n-1)!} \|f^{(n)}\|_\infty \int_0^1 | K_{\xi^*}(t)|\, dt.$$

Now $\sum_{i=1}^{n+r} a_i^* P(x_i) = \epsilon \| P \|_\infty \sum_{i=1}^{n+r} | a_i^* |$, where $\epsilon = +1$ or -1, fixed, and $\int_0^1 K_{\xi^*}(t) P^{(n)}(t)\, dt = \lambda \| P^{(n)} \|_\infty \int_0^1 | K_{\xi^*}(t)|\, dt$, where $\lambda = +1$ or -1, fixed. To prove the theorem, it is necessary that we show that $\epsilon = \lambda$.

Assume that $\operatorname{supp} K_{\xi^*}(t) = (x_{i_1}, x_{i_2})$. For $t = x_{i_2} - \delta$, $\delta > 0$, small, $K_{\xi^*}(t) = -a_{i_2}^*(x_{i_2} - t)_+^{n-1}$, and therefore

$$\operatorname{sgn} K_{\xi^*}(t) = -\operatorname{sgn} a_{i_2}^* \qquad \text{for} \quad t \in (x_{i_2} - \delta, x_{i_2}). \tag{6.11}$$

Assume $P(x_i)(-1)^{i+n+r} > 0, i = 1,..., n + r$. Thus, $\operatorname{sgn} P(x_{i_2}) = (-1)^{i_2+n+r}$. Since $P(x)$ has opposite orientation, $P^{(n)}(t)(-1)^{i+r} > 0$ for $\eta_i < t < \eta_{i+1}$, $i = 0, 1,..., r + 1$, where $\eta_0 = 0$, $\eta_{r+2} = 1$. From (6.10), $\eta_{i_2-n+1} < x_{i_2} \leqslant \eta_{i_2-n+2}$. Hence, for $t \in (x_{i_2} - \delta, x_{i_2})$, $P^{(n)}(t)(-1)^{i_2-n+r+1} > 0$. It therefore follows from (6.11) that $\epsilon = \lambda$. The theorem is proved.

$\mathscr{P}(\sigma)$ contains perfect splines of the above form as well as Zolotarev perfect splines. For $\sigma \in (\sigma_r, \sigma_{r+1})$, the Zolotarev spline has $r + 1$ knots and $n + r + 1$ points of equioscillation. Any other perfect spline $P(x)$ satisfying these properties is such that $P(x) = \pm Z(x; \sigma)$ or $P(x) = \pm Z(1 - x; \sigma)$ (see Theorem 2.4). Let us assume the normalization $Z(1; \sigma) = 1$ and $Z^{(n)}(1; \sigma) = \sigma$, which uniquely determines $Z(x; \sigma)$. Let $0 \leqslant x_1 < \cdots < x_{n+r} < x_{n+r+1} = 1$ denote the points of equioscillation of $Z(x; \sigma)$ and let $\{\eta_i\}_{i=1}^{r+1}$, $0 < \eta_1 < \cdots < \eta_{r+1} < 1$ denote its knots. Then, as in (5.2), $x_i < \eta_i < x_{i+n-1}$, $i = 1,..., r + 1$. If we discard the point $x_{n+r+1} = 1$, then we see that we are in the situation where the previous analysis is applicable.

If $\sigma = \sigma_{r+1}$, then $Z(x; \sigma_{r+1}) = P_{r+1}(x)$. $P_{r+1}(x)$ has $n + r + 2$ points of equioscillation, one at zero and one at one, and $r + 1$ knots. In order to apply the previous analysis, we delete the points of equioscillation at 0 and 1.

Note that the point (or points if $\sigma = \sigma_{r+1}$) of equioscillation which are deleted are chosen in order that the perfect spline have opposite orientation with respect to the remaining points of equioscillation.

Thus it follows that

THEOREM 6.2. *For each $\sigma > 0$ and k, $1 \leqslant k \leqslant n-2$ and for each $P \in \mathscr{P}(\sigma)$, there exists at least one point $\xi \in (0, 1)$ such that* $\max |f^{(k)}(\xi)|$, *over $f \in W_\infty^{(n)}(\sigma)$, is attained by $P(x)$.*

The maximization problem (6.2) where $\xi = 0$ or 1, i.e., an endpoint, has been considered by Karlin [11]. If $\sigma = \sigma_r$, then $P_r(x)$ maximizes $|f^{(k)}(1)|$ (and $|f^{(k)}(0)|$) for all $k = 1,..., n-1$. If $\sigma_r < \sigma < \sigma_{r+1}$, then $Z(x; \sigma)$ with the above normalization maximizes $|f^{(k)}(1)|$ for $k = 1,..., n-1$. At the endpoint zero $Z(1-x; \sigma)$ is, of course, a maximizing function. There are various methods of proof of this result. We refer the reader to Karlin [11] for a proof which is also given via a numerical differentiation formula.

In the above analysis, we did not consider the case $k = n-1$. This was, in the main, due to various technical difficulties brought about by the lack of continuity of $s^{(n-1)}(x)$ and $K_\xi(t)$ for $k = n-1$. Below, we reconsider the relevant portions of the preceding analysis in order to prove

THEOREM 6.3. *Let $P \in \mathscr{P}(\sigma)$. Then, for any $f \in W_\infty^{(n)}(\sigma)$, we have $|f^{(n-1)}(\eta)| \leqslant |P^{(n-1)}(\eta)|$ where η is any knot of $P(x)$.*

We shall only consider the case where $P(x)$ is a perfect spline with $r+1$ knots, $n+r$ points of equioscillation, and opposite orientation. As indicated above, the analysis remains valid for the Zolotarev perfect splines with the previously considered modifications.

From Proposition 6.1, $s^{(n-1)}(x)$ is a nonzero constant on each (η_i, η_{i+1}), $i = 0, 1,..., r+1$, where $\eta_0 = 0$, $\eta_{r+2} = 1$, and $\{\eta_i\}_{i=1}^{r+1}$ are the knots of $P(x)$, and $s^{(n-1)}(\eta_i-)\, s^{(n-1)}(\eta_i+) < 0$, $i = 1,..., r+1$. Let $L(\xi)$ denote the $(n+r+1) \times (n+r+1)$ matrix

$$\begin{bmatrix} 1 & \cdots & 1 & 0 \\ x_1 & \cdots & x_{n+r} & 0 \\ \vdots & & \vdots & \vdots \\ x_1^{n-2} & \cdots & x_{n+r}^{n-2} & 0 \\ x_1^{n-1} & \cdots & x_{n+r}^{n-1} & (n-1)! \\ (x_1-\eta_1)_+^{n-1} & \cdots & (x_{n+r}-\eta_1)_+^{n-1} & (n-1)!(\xi-\eta_1)_+^0 \\ \vdots & & \vdots & \vdots \\ (x_1-\eta_{r+1})_+^{n-1} & \cdots & (x_{n+r}-\eta_{r+1})_+^{n-1} & (n-1)!(\xi-\eta_{r+1})_+^0 \end{bmatrix},$$

where $\{x_i\}_{i=1}^{n+r}$ are the points of equioscillation of $P(x)$. Since $x_i < \eta_i <$

x_{i+n-1}, $i = 1,\ldots, r+1$, the $(n+r) \times (n+r)$ principal submatrix of $L(\xi)$ obtained by deleting the last row and column is nonsingular. Moreover,

$$(b_0, \ldots, b_{n-1}, c_1, \ldots, c_{r+1})\, L(\xi) = (0, \ldots, 0, s^{(n-1)}(\xi)).$$

Thus, if $s^{(n-1)}(\xi) \neq 0$, then $\det L(\xi) \neq 0$. It then follows, since $s^{(n-1)}(\eta_j -)\, s^{(n-1)}(\eta_j +) < 0$, $j = 1,\ldots, r+1$, that $\det L(\eta_j -) \cdot \det L(\eta_j +) < 0$, $j = 1,\ldots, r+1$. Fix j, $1 \leqslant j \leqslant r+1$, and let $\zeta_i = \eta_i$, $i = 1,\ldots, j-1$, $\zeta_i = \eta_{i+1}$, $i = j,\ldots, r$. Construct the numerical differentiation formula

$$f^{(n-1)}(\eta_j) = \sum_{i=1}^{n+r} a_i f(x_i) + \frac{1}{(n-1)!} \int_0^1 K(t) f^{(n)}(t)\, dt$$

such that $K(\zeta_i) = 0$, $i = 1,\ldots, r$, where $\{x_i\}_{i=1}^{n+r}$ and $\{\zeta_i\}_{i=1}^{r}$ are as stipulated above. This construction is equivalent to solving the linear system

$$\begin{bmatrix} 1 & \cdots & 1 \\ x_1 & & x_{n+r} \\ \vdots & & \vdots \\ x_1^{n-2} & \cdots & x_{n+r}^{n-2} \\ x_1^{n-1} & \cdots & x_{n+r}^{n-1} \\ (x_1 - \zeta_1)_+^{n-1} & \cdots & (x_{n+r} - \zeta_1)_+^{n-1} \\ \vdots & & \vdots \\ (x_1 - \zeta_r)_+^{n-1} & \cdots & (x_{n+r} - \zeta_r)_+^{n-1} \end{bmatrix} \begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{n+r} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ (n-1)! \\ (n-1)!(\eta_j - \zeta_1)_+^0 \\ \vdots \\ (n-1)!(\eta_j - \zeta_r)_+^0 \end{bmatrix}. \tag{6.12}$$

Since $x_i < \eta_i < x_{i+n-1}$, it follows that $x_i < \zeta_i < x_{i+n}$ for $i = 1,\ldots, r$ and these are explicitly the conditions necessary to insure the nonsingularity of the above matrix. Now,

$$K(t) = \det \begin{bmatrix} 1 & \cdots & 1 & 0 \\ x_1 & \cdots & x_{n+r} & 0 \\ \vdots & & \vdots & \vdots \\ x_1^{n-2} & \cdots & x_{n+r}^{n-2} & 0 \\ x_1^{n-1} & \cdots & x_{n+r}^{n-1} & (n-1)! \\ (x_1 - \zeta_1)_+^{n-1} & \cdots & (x_{n+r} - \zeta_1)_+^{n-1} & (n-1)!(\eta_j - \zeta_1)_+^0 \\ \vdots & & \vdots & \vdots \\ (x_1 - \zeta_r)_+^{n-1} & \cdots & (x_{n+r} - \zeta_r)_+^{n-1} & (n-1)!(\eta_j - \zeta_r)_+^0 \\ (x_1 - t)_+^{n-1} & \cdots & (x_{n+r} - t)_+^{n-1} & (n-1)!(\eta_j - t)_+^0 \end{bmatrix} E,$$

where E is the reciprocal of the determinant of the matrix given in (6.12). Thus $K(t) = (n-1)!(\eta_j - t)_+^0 - \sum_{i=1}^{n+r} a_i(x_i - t)_+^{n-1}$ satisfies $K(\zeta_i) = 0, i = 1,..., r$, and $K(\eta_j+) K(\eta_j-) < 0$.

We wish to prove that $a_i(-1)^i \gamma \geqslant 0$, $i = 1,..., n + r$, where $\gamma = +1$ or -1, fixed, and $K(t)(-1)^i \delta \geqslant 0$ for $\eta_{i-1} < t < \eta_i$, $i = 1,..., r + 2$, where $\eta_0 = 0$, $\eta_{r+2} = 1$, and $\delta = +1$ or -1, fixed. As in the proof of Proposition 6.3, the analysis is divided into various cases. We shall herein consider only the analog of Case 1, Subcase 1.1 of Proposition 6.3. The remaining cases follow in a similar manner. Hence, let us assume that $K(t) \not\equiv 0$ on any subinterval of (x_1, x_{n+r}) and $\eta_j \neq x_i$, $i = 2,..., n + r - 1$.

By Theorem 5.2,

$$\tilde{Z}_K(x_1, \eta_j) \leqslant S^-\left(\sum_{i=2}^{n+r} a_i^*,..., \sum_{i=l}^{n+r} a_i^*\right) + S^-(K(x_1),..., K^{(n-1)}(x_1+)) - S^+(K(\eta_j-),..., K^{(n-1)}(\eta_j-))$$

and

$$\tilde{Z}_K(\eta_j, x_{n+r}) \leqslant S^-\left(\sum_{i=l}^{n+r} a_i^*,..., a_{n+r}^*\right) + S^-(K(\eta_j+),..., K^{(n-1)}(\eta_j+)) - S^+(K(x_{n+r}),..., K^{(n-1)}(x_{n+r}-)),$$

for some l, $2 \leqslant l \leqslant n + r$. Thus

$$r \leqslant \tilde{Z}_K(x, \eta_j) + \tilde{Z}_K(\eta_j, x_{n+r}) \leqslant S^-\left(\sum_{i=2}^{n+r} a_i^*,..., a_{n+r}^*\right) - (n-1) + S^-(K(\eta_j+),..., K^{(n-1)}(\eta_j+)) - S^+(K(\eta_j-),..., K^{(n-1)}(\eta_j-)),$$

implying

$$1 \leqslant S^-(K(\eta_j+),..., K^{(n-1)}(\eta_j+)) - S^+(K(\eta_j-),..., K^{(n-1)}(\eta_j-)).$$

Since $K^{(i)}(\eta_j-) = K^{(i)}(\eta_j+)$, $i = 1,..., n-1$,

$$S^-(K(\eta_j+),..., K^{(n-1)}(\eta_j+)) - S^+(K(\eta_j-),..., K^{(n-1)}(\eta_j-)) \leqslant 1.$$

Equality in the above equations implies $K(\eta_j+) K(\eta_j-) < 0$, $K(t)$ changes sign at ζ_i, $i = 1,..., r$ and nowhere else in $(x_1, \eta_j) \cup (\eta_j, x_{n+r})$, and $a_i^* a_{i-1}^* < 0$, $i = 1,..., n + r - 1$.

The remaining analysis is totally analogous to that given for the case $1 \leqslant k \leqslant n - 2$.

Remark 6.1. Note that the results of this section are independent of Sections 3 and 4, and Theorem 5.1.

Remark 6.2. The results of this paper extend, mutatis mutandis, to the class of functions

$$f(x) = \sum_{i=0}^{n-1} a_i u_i(x) + \int_0^1 K(x, t)(Lf)(t)\, dt,$$

where L is an nth-order differential operator of Pólya type W (totally disconjugate) on $[0, 1]$, $\{u_i(x)\}_{i=0}^{n-1}$ is a basis of solutions of $Lf = 0$, and $K(x, t)$ is the fundamental solution for $Lf = 0$ obtained by zero initial data at zero, see Karlin [8–11]. The restriction $\| f^{(n)} \|_\infty \leqslant \sigma$ is here replaced by $\| Lf \|_\infty \leqslant \sigma$.

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